Week 3 Banach space

Let X be a metric space.

Recall:

① A sequence (x_n) in X is said to be convergent if $\exists y \in X$ such that $\lim_{n\to\infty} d(x_n, y) = 0$

Write $y = \lim_{n\to\infty} x_n$.

Such a y is unique if it exists

② (x_n) is said to be Cauchy if $\forall \varepsilon > 0$, $\exists N$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n > N$

③ X is said to be complete if every Cauchy sequence in it is convergent

Rmk

$$\text{Convergent sequence} \xRightarrow{\text{Always}} \text{Cauchy sequence}$$

$$\text{Convergent sequence} \xLeftarrow[\text{true if } X \text{ is complete}]{} \text{Cauchy sequence}$$

Thm 1.4.7

Let X be a complete metric space.

$M \subset X$ is a subset

M is closed in X $\iff$ M is complete

<u>Pf</u> ($\Rightarrow$ part) Suppose M is closed in X

Let (x_n) be a Cauchy sequence in M

then (x_n) is a Cauchy sequence in X

X is complete $\Rightarrow$ $\exists\, y \in X$ such that

$$\lim_{n\to\infty} x_n = y$$

M is closed, $x_n \in M$

$\Rightarrow$ $y = \lim_{n\to\infty} x_n \in \overline{M} = M$

$\Rightarrow$ (x_n) is a convergent sequence in M

(with limit y in M)

($\Leftarrow$ part) Suppose M is complete

Want to prove $\overline{M} = M$

Suppose $y \in \overline{M} \subset X$

$\Rightarrow$ $\exists$ a sequence $x_n \in M$ such that

$$\lim_{n\to\infty} x_n = y \in X$$

$\Rightarrow$ x_n is a convergent sequence in X

$\Rightarrow$ x_n is Cauchy sequence

M is complete ; $x_n \in M$

$\Rightarrow$ $y = \lim_{n\to\infty} x_n \in M$

$\Rightarrow$ $\overline{M} \subseteq M$

Also, $M \subseteq \overline{M}$ (always true) $\Rightarrow M = \overline{M}$

$\Rightarrow$ M is closed in X

Thm 1.5-2, 1.5-4

l^p, l^∞ are complete Normed space

Pf for l^p, $(1 \le p < \infty)$

Let (x_n) be a Cauchy sequence in l^p

$x_n = (x_{n,1}, x_{n,2}, x_{n,3}, \cdots \cdots)$

$x_1 = (x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, \cdots\cdots)$

$x_2 = (x_{2,1}, x_{2,2}, x_{2,3}, x_{2,4}, \cdots\cdots)$

$x_3 = (x_{3,1}, x_{3,2}, x_{3,3}, x_{3,4}, \cdots\cdots)$

$x_4 = (x_{4,1}, x_{4,2}, x_{4,3}, x_{4,4}, \cdots)$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$y = (y_1, y_2, y_3, y_4, \cdots)$

Strategy ① Guess the limit of (x_n)

② Show that it is really the limit

(x_n) is Cauchy

$\Rightarrow \forall \varepsilon > 0, \exists N > 0$ such that $\forall m, n > N$,

$$d(x_n, x_m) = \|x_n - x_m\|_p = \left(\sum_{j=1}^{\infty} |x_{n,j} - x_{m,j}|^p\right)^{\frac{1}{p}} < \varepsilon \quad (*)$$

$\Rightarrow$ For any j (fixed)

$$|x_{n,j} - x_{m,j}| \le \left(\sum_{k=1}^{\infty} |x_{n,k} - x_{m,k}|^p\right)^{\frac{1}{p}} < \varepsilon$$

$\Rightarrow (x_{1,j}, x_{2,j}, x_{3,j}, \cdots)$ is Cauchy

$\mathbb{R}$ is complete $\Rightarrow$ the sequence is convergent

Let $y_j = \lim_{m \to \infty} x_{m,j}$ and $y = (y_1, y_2, y_3, \cdots)$

Next, want to show

① $y \in \ell^p$

② $\lim_{n\to\infty} x_n = y$ in ℓ^p-norm

From ⊛, for any k and $m,n > N$

$$\sum_{j=1}^{k} |x_{n,j} - x_{m,j}|^p < \varepsilon^p$$

let $m \to \infty$

$$\sum_{j=1}^{k} |x_{n,j} - y_j|^p \le \varepsilon^p$$

let $k \to \infty$

$$\sum_{j=1}^{\infty} |x_{n,j} - y_j|^p \le \varepsilon^p < \infty \quad \text{**}$$

$\Rightarrow \quad x_n - y \in \ell^p$

But ℓ^p is a vector space (From Minkowski inequality)

$$\Rightarrow \quad y = \underset{\substack{\Uparrow \\ \ell^p}}{x_n} - \underset{\substack{\Uparrow \\ \ell^p}}{(x_n - y)} \in \ell^p \quad \Leftarrow$$

** $\Rightarrow \quad d(x_n, y) \le \varepsilon$ for $n > N$

ε can be arbitrarily small

$\Rightarrow \quad \lim_{n\to\infty} x_n = y$

Let X be metric space

Thm The following are equivalent (TFAE)

① Any open cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of X has a finite cover

② Any sequence in X has a convergent subsequence

Rmk ① $\{J_\alpha\}_{\alpha\in\Lambda}$ is open cover means

- Λ is index set
- $X = \bigcup_{\alpha\in\Lambda} J_\alpha$, and each J_α is open in X

Usually

$$\underset{\text{compact}}{①} \overset{\text{for metric space}}{\Longleftrightarrow} \underset{\text{Sequential compact}}{②}$$

Indeed, textbook uses ② as definition of compactness

Important fact Let X be metric space, A be a subset of X

① Let Y be a metric space, $f: X \to Y$ be continuous. Also, A is compact then $f(A)$ is also compact

2. If A is compact, then A is closed and bounded

3. If $X = \mathbb{R}^n$ or $\mathbb{C}^n$

   A is compact $\Longleftrightarrow$ A is closed and bounded

Next: Show that finite dimensional normed spaces are complete

<u>Lemma 2.4-1</u> $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Let X be a normed space,

$S = \{x_1, x_2, \cdots, x_n\}$ be linear independent subset of X

Then $\exists\, c > 0$ such that

$$\underbrace{\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\|}_{\text{norm of a vector}} \geq c\,(\underbrace{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|}_{\text{magnitude of coefficients}})$$

Meaning: If coefficients are not small then norm is not small

<u>Pf</u> (Different from textbook)

Assume $\mathbb{F} = \mathbb{R}$ (similar for $\mathbb{C}$)

<u>Pf</u> Let $S = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$

<u>Case 1:</u> $S = 0$

$\Rightarrow |\alpha_1| = |\alpha_2| = \cdots = |\alpha_n| = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 \cdots = \alpha_n = 0$

In this case L.H.S. $= 0 =$ R.H.S.

$\Rightarrow$ inequality holds for any $c > 0$

<u>Case 2:</u> $S = 1$

the inequality becomes:

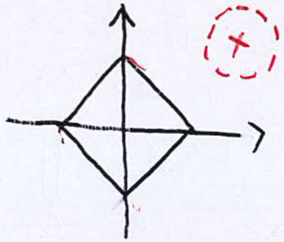
$$\|\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n\| \geq c$$

Define $f : \mathbb{R}^n \longrightarrow X$

$$f(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n$$

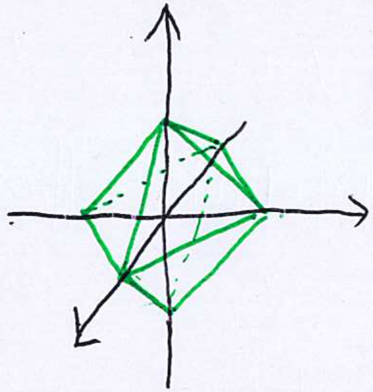
Let $Y = \{(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{R}^n : |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| = 1\}$

$n=2$



$|x| + |y| = 1$

$n=3$



$|x| + |y| + |z| = 1$

Note:

① Y is compact $\left(\begin{array}{l}\text{closed} \\ + \text{ bounded}\end{array}\right)$

② f is continuous

$\Rightarrow f(Y)$ is compact

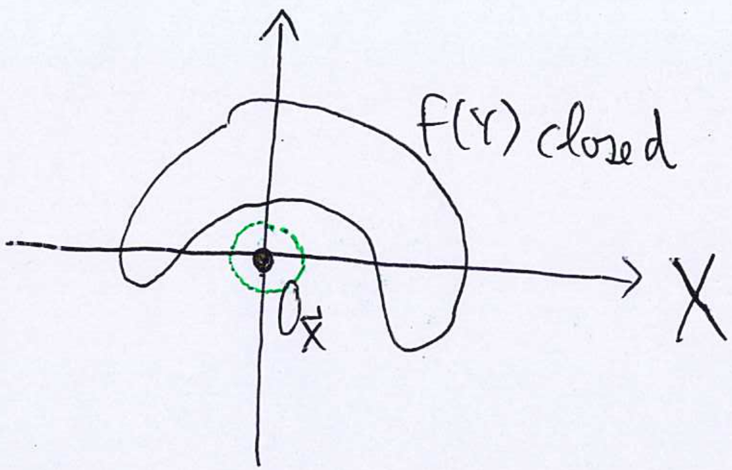
$\Rightarrow f(Y)$ is closed in X

Also, S is linearly independent

$\therefore \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \neq \vec{0}_X$

unless $\alpha_1 = \alpha_2 = \alpha_3 = \cdots \alpha_n = 0$

$\Rightarrow \vec{0}_X \notin f(Y)$



$f(Y)$ is closed

$\Rightarrow \exists c > 0$ such that $B(\vec{0}_X, c) \cap f(Y) = \phi$

But $B(\vec{0}_X, c) = \{x \in X : \|x - \vec{0}_X\| < c\}$

$= \{x \in X : \|x\| < c\}$

$\Rightarrow \forall x \in f(Y), \ \|x\| \geq c$

$f(Y) = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n : \sum_i |\alpha_i| = 1\}$

$\Rightarrow \|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\| \geq c$

for all $|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| = 1$

$\Rightarrow$ Case 2

<u>Case 3</u>: $s \neq 0$, ie $s > 0$

let $\beta_i = \dfrac{\alpha_i}{s}$

Then

$|\beta_1| + |\beta_2| + \cdots + |\beta_n|$

$= \left|\dfrac{\alpha_1}{s}\right| + \left|\dfrac{\alpha_2}{s}\right| + \cdots + \left|\dfrac{\alpha_n}{s}\right|$

$= \dfrac{1}{s}\left(|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|\right)$

$= \dfrac{1}{s} \cdot s = 1$

Case 2 $\Rightarrow \|\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n\| \geq c$

$\Rightarrow \left\|\dfrac{\alpha_1}{s} x_1 + \dfrac{\alpha_2}{s} x_2 + \cdots + \dfrac{\alpha_n}{s} x_n\right\| \geq c$

$\Rightarrow \|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\| \geq cs$

where $cs = c\left(|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|\right)$

$\Rightarrow$ Case 3

<u>Thm 2.4-2</u> Every finite dimensional subspace Y of a normed space X is complete

In particular, every finite dimension normed space is complete

<u>Pf</u> Let Y have basis $\{x_1, x_2, \cdots, x_n\}$

Suppose (y_m) be a Cauchy sequence in Y

Let

$y_m = \alpha_{m,1} x_1 + \alpha_{m,2} x_2 + \cdots + \alpha_{m,n} x_n$

Want to show for each $1 \le j \le n$ coefficients $(\alpha_{m,j})$ is a Cauchy sequence

$\forall \varepsilon > 0$, $\exists N$ such that if $m, r > N$ then $\|y_m - y_r\| < \varepsilon$

Lemma 2.4-1 $\Rightarrow \exists c > 0$ s.t.

$$\varepsilon > \|y_m - y_r\| = \left\| \sum_{j=1}^{n} (\alpha_{m,j} - \alpha_{r,j}) x_j \right\| \ge c \left(\sum_{j=1}^{n} |\alpha_{m,j} - \alpha_{r,j}| \right)$$

$\Rightarrow$ For each i

$$|\alpha_{m,i} - \alpha_{r,i}| \le \sum_{j=1}^{n} |\alpha_{m,j} - \alpha_{r,j}| < \frac{\varepsilon}{c}$$

$\Rightarrow (\alpha_{1,i}, \alpha_{2,i}, \alpha_{3,i}, \cdots)$ is Cauchy

Also, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is complete

$\Rightarrow \lim_{m \to \infty} \alpha_{m,i} = \beta_i$ exists $\forall i$

Let $y = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n \in Y$

Then

$$0 \le \|y_m - y\| = \left\| \sum_{j=1}^{n} (\alpha_{m,j} - \beta_j) x_j \right\| \le \sum_{j=1}^{n} \|(\alpha_{m,j} - \beta_j) x_j\| = \sum_{j=1}^{n} |\alpha_{m,j} - \beta_j| \, \|x_j\|$$

Take $m \to \infty$

R.H.S $\to 0$

Sandwich theorem

$\Rightarrow \lim_{m\to\infty} \| y_m - y \| = 0$

$\Rightarrow \lim_{m\to\infty} y_m = y \in Y$

$\therefore$ The Cauchy sequence is convergent in Y

$\Rightarrow Y$ is complete

HW 1 Hint

<u>Q3a</u> $p < q$ (Want to show $\ell^p \subset \ell^q$)

① Let $\vec{x} = (x_1, x_2, \cdots) \in \ell^p$

$\therefore \sum_{i=1}^{\infty} |x_i|^p < \infty$

$\Rightarrow \lim_{i\to\infty} |x_i|^p = 0$, i.e. $x_i \to 0$

In particular $|x_i| < 1$ for large i

② $|x_i|^q < |x_i|^p$ if $|x_i| < 1$

$\sum_{i=1}^{\infty} |x_i|^p = |x_1|^p + |x_2|^p + \cdots + |x_i|^p + \cdots$

$\sum_{i=1}^{\infty} |x_i|^q = |x_1|^q + |x_2|^q + \cdots + |x_i|^q + \cdots$

Q3b Show $l^p \subset l^q$ is proper subset

ie. $l^p \subset l^q$ and $l^p \neq l^q$

ie

$\exists$ element $\vec{x} \in l^q$ but not in l^p

In class:

$\vec{x} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots) \in l^p, p>1$

$\notin l^1$

Hint: Modify